Lecture #5

Optimization for multivariate functions

- We are now familiar with 1^{st}-order necessary condition for finding critical points of multivariate functions.

$\downarrow$ Compute all partial derivatives

and set the entire gradient vector to zero in order to find the critical point.

- Suppose we need to optimize the function $f(x,y)$

Function of two variables.

1^{st}- order necessary condition gives all the critical points.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{bmatrix} = 0$$ gives all critical points.

Let us say (x^*, y^*) is one such stationary point or critical point

$$f_x := \frac{\partial f}{\partial x}, \quad f_y := \frac{\partial f}{\partial y}, \quad f_{xy} := \frac{\partial^2 f}{\partial y \partial x}, \quad f_{yx} := \frac{\partial^2 f}{\partial x \partial y}$$

If the function is continuous, then $f_{xy} = f_{yx}$.

* If $f_{xx} > 0$ and determinant D of the matrix, $D > 0$

$$\begin{bmatrix} f_{xx} & f_{xy} \\ \\ f_{yx} & f_{yy} \end{bmatrix}_{(x^*, y^*)}$$

$\Downarrow$

then (x^*, y^*) is a local minimum.

* If $f_{xx} < 0$ and $D > 0$, then (x^*, y^*) is a local maximizer

* If $D < 0$, then (x^*, y^*) is a saddle point.

Ex: Find all critical points of the function

$$f(x,y) = x^2y - 2xy^2 + 3xy + 4$$

$$f_x = 2xy - 2y^2 + 3y$$
$$f_y = x^2 - 4xy + 3x$$

For critical points, $f_x = f_y = 0$

$$\Rightarrow f_x = 0 \Rightarrow y(2x - 2y + 3) = 0$$
$$f_y = 0 \Rightarrow x(x - 4y + 3) = 0$$

These two equations must be satisfied simultaneously. There are four possibilities.

1. $(x=0, y=0)$
2. $(x=0, 2x-2y+3=0) \Rightarrow (x=0, y=3/2)$
3. $(y=0, x-4y+3=0) \Rightarrow (x=-3, y=0)$
4. $(2x-2y+3=0, x-4y+3=0) \Rightarrow (x=-1, y=1/2)$

Let us evaluate the function at these critical points.

$$f(0,0) = 4 \; ; \; f(0,3/2) = 4 \; ; \; f(-3,0) = 4 \; ; \; f(-1, 1/2) = 7/2$$

Ex: A more practical optimization problem

Find three positive numbers whose sum is 100 and whose product is maximum.

$$\max \; f(x,y,z) = xyz$$

$$\text{s.t. } x + y + z = 100.$$

This is a constrained optimization problem that can be converted into unconstrained " " using

$$z = 100 - x - y.$$

Therefore, the optimization problem is

$$\max\ g(x,y) = xy(100-x-y)$$

Let us find the critical points of $g(x,y)$.

$$g_x = 0 \Rightarrow y(100-2x-y) = 0$$
$$g_y = 0 \Rightarrow x(100-x-2y) = 0$$

Again, we get four critical points.

1. $(x=0, y=0)$
2. $(x=0, y=100)$
3. $(x=100, y=0)$

$\rightarrow$ unacceptable as the numbers are positive.

4. $(x=100/3, y=100/3)$ $\rightarrow$ This is the only feasible candidate sol^n. We still need to make sure that the function gets maximized at $\left(x^* = y^* = \frac{100}{3}\right)$

Recall $g_x = 100y - 2xy - y^2$
and $g_y = 100x - x^2 - 2xy$

$$\Rightarrow g_{xx} = -2y;\ g_{yy} = -2x;\ g_{xy} = g_{yx} = 100 - 2x - 2y$$

$$D = \det \begin{bmatrix} -\frac{200}{3} & -100/3 \\ -\frac{100}{3} & -\frac{200}{3} \end{bmatrix}$$

$g_{xx} < 0$ and $D > 0$

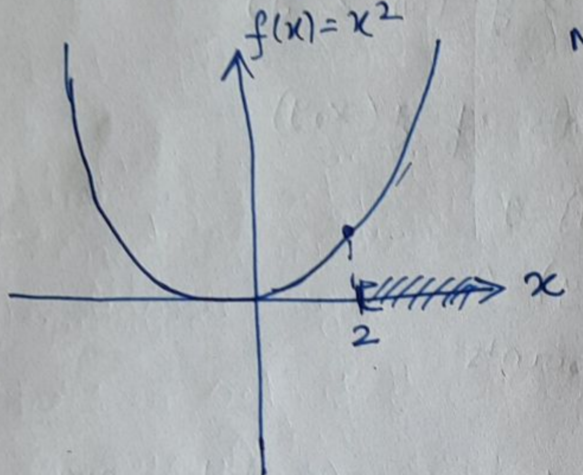
$$\Rightarrow \left(x^* = y^* = \frac{100}{3}\right) \text{ is a local maximizer.}$$

$\therefore$ Solution to the original problem.

$$x^* = y^* = z^* = \frac{100}{3}$$

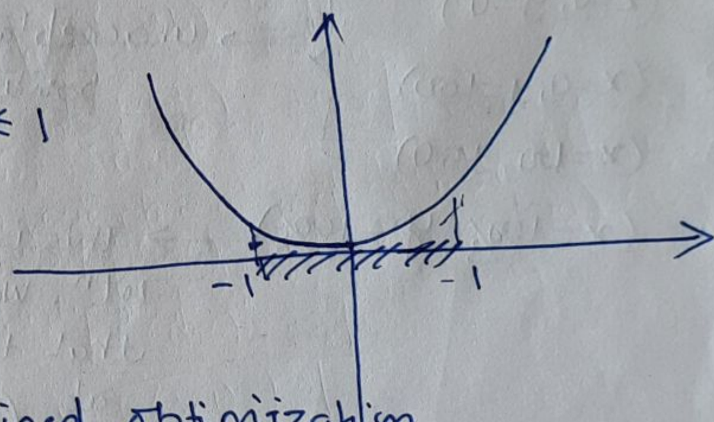
Optimization of a function on a Region
(Constrained Optimization)

Motivation: We know $f(x) = x^2$ gets minimized at $x = 0$, but what if we consider a slightly different problem.



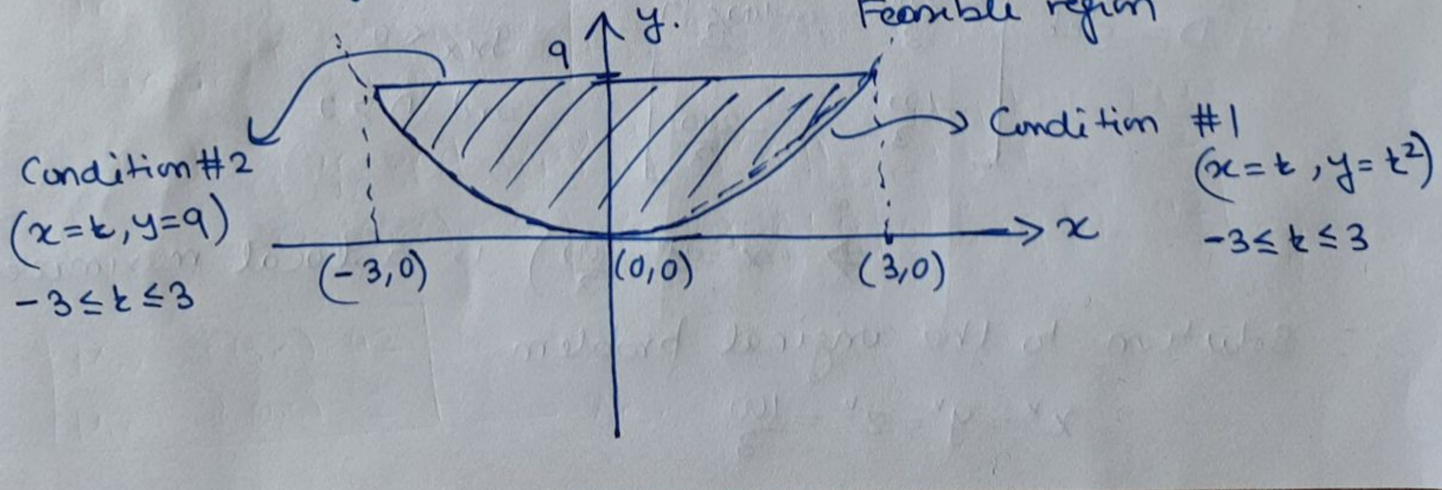
Min $f(x)$
s.t. $x \geq 2$
↓
The function achieves a minimum at the boundary of the feasible region, i.e. at $x = 2$

Min $f(x) = x^2$
s.t. $-1 \leq x \leq 1$
↓
Minimum is achieved at the stationary point.



- when we deal with constrained optimization we need to look beyond the stationary points as well, In particular, we need to evaluate function at the boundary of the feasible region.

Ex: Find absolute max and min of $f(x,y) = xy - 3x$ in the region with $x^2 \leq y \leq 9$.



- Critical point $f_x = y-3$, $f_y = x$ $\Big\}$ $\Rightarrow$ Single critical point $\underbrace{(0,3)}_{\text{Inside the feasible region}}$

$\boxed{f(0,3) = 0}$

- Next we analyze the boundary of the region.

$x = t,\ y = t^2 \quad -3 \leq t \leq 3 \qquad f(t, t^2) = t^3 - 3t$

Two critical points. $(-1,1)$ and $(1,1)$

$\boxed{f(-1,1) = 2}$ and $\boxed{f(1,1) = -2}$

- Now in the region, $x = t,\ y = 9 \quad -3 \leq t \leq 3$

$f(t,9) = 6t$ has no critical points.

We only have boundary points $(-3,9)$ and $(3,9)$

$\boxed{f(-3,9) = -18}$ and $\boxed{f(3,9) = 18}$

- Full list of points to analyze,

$(0,3)$, $(-1,1)$, $(1,1)$, $(-3,9)$ and $(3,9)$

$\boxed{\text{Max is } 18}$ and $\boxed{\text{Min is } -18}$

This example has both objective function and constraints that are nonlinear in x and y. → Nonlinear programming

- When the objective function and constraints are linear in decision variables, we call it a linear program.

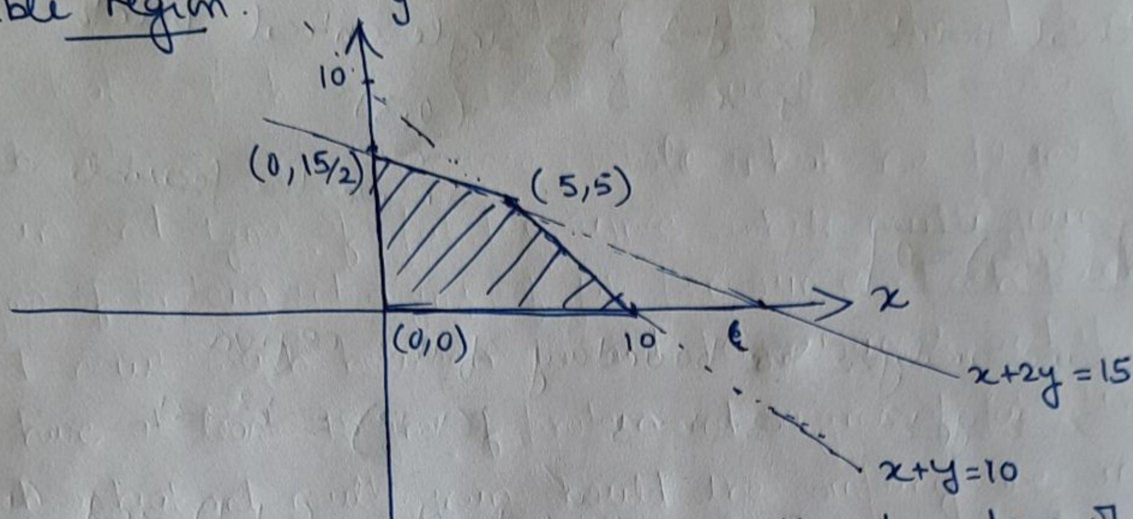
Ex: Linear Program:

Find min and max of $f(x,y) = 2 + 3x + 5y$.

s.t. $x \geq 0,\ y \geq 0,\ x + y \leq 10,\ x + 2y \leq 15$

- Linear programs do not have any critical points.

Feasible region:



We need to evaluate the function at the boundary of the feasible region

1. Case 1: $x=0$, $y=t$ $\quad 0 \le t \le \frac{15}{2}$

$$f(0,t) = 2+5t \Rightarrow \boxed{f(0,0)=2}$$
$$\boxed{f\left(0,\tfrac{15}{2}\right) = 39.5}$$

2. Case 2: $x=t$, $y=0$, $\quad 0 \le t \le 10$

$$f(t,0) = 2+3t \Rightarrow \boxed{f(0,0)=2}$$
$$\boxed{f(10,0) = 32}$$

3. Case 3: $x=t$, $y=10-t$ $\quad [\because x+y=10]$

$$5 \le t \le 10$$

$$f(t,10-t) = 2+3t+50-5t = 52-2t$$

$$\boxed{f(5,5) = 42} \qquad \boxed{f(10,0) = 32}$$

4. Case 4: $x=t$, $y=\frac{15-t}{2}$, $\quad 0 \le t \le 5$

$$f\left(t,\tfrac{15-t}{2}\right) = 2+3t+\tfrac{5}{2}(15-t)$$
$$= 2+3t+\tfrac{75}{2}-\tfrac{5t}{2} = \tfrac{79}{2}+\tfrac{t}{2} = \tfrac{79+t}{2}$$

$$\cancel{\Rightarrow f(0,0)=0} \quad \boxed{f\left(0,\tfrac{15}{2}\right)=39.5} \quad \boxed{f(5,5)=42}$$

$$\boxed{\begin{array}{l}\text{Max. } f(5,5)=42 \\ \text{Min. } f(0,0)=2\end{array}}$$ —— Ans.

Ex: A bread manufacturing company makes its bread from whole-wheat, which costs 25 cents per ounce, and multi-grain, which costs 20 cents per ounce. Whole wheat has 10 gms of protein and 4 gms of fat per ounce, while multi-grain has 5 gms of protein and 8 gms of fat per ounce. Each package of food must weigh between 10 and 16 ~~gms~~ ounces, and it must also have at least 95 gms of protein and 80 gms of fat. How much wheat and multi-grain should the company use in each package to minimize the total cost while also satisfying these requirements?

A: Let 'w' and 'm' denote the amount of wheat and multi-grain (in ounces).

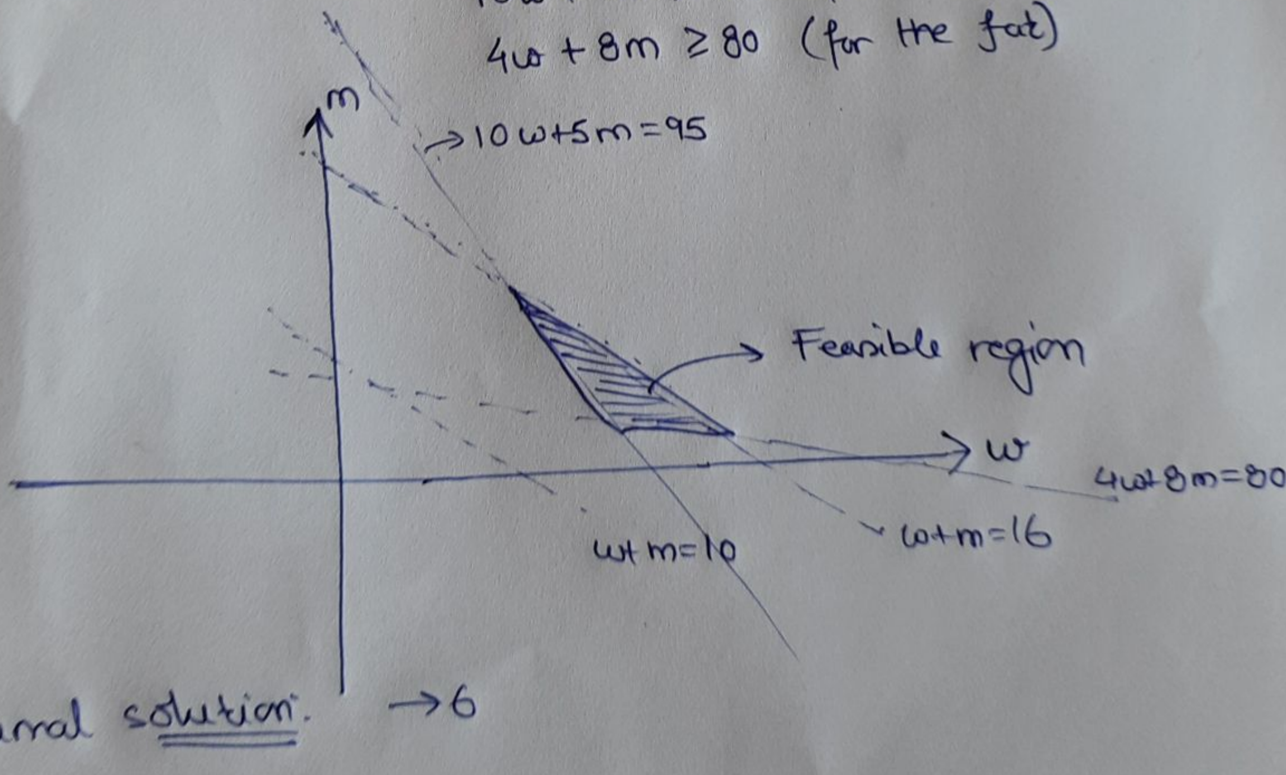
We wish to minimize, $f(w,m) = 25w + 20m$

subject to $w \geq 0,\ m \geq 0$ — (positivity constraint)

$10 \leq w+m \leq 16$ (weight constraint)

$10w + 5m \geq 95$ (for the protein)

$4w + 8m \geq 80$ (for the fat)



Optimal solution: → 6